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# COMPATIBLE MAPS AND COMPATIBLE MAPS OF TYPES ($\alpha$) AND ($\beta$) IN INTUITIONISTIC FUZZY METRIC SPACES

**Abstract.** In this paper, we first formulate the definition of compatible maps and compatible maps of types ($\alpha$) and ($\beta$) in intuitionistic fuzzy metric spaces and give some relations between the concepts of compatible maps and compatible maps of types ($\alpha$) and ($\beta$).

## Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [25]. Since then many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [5], Erceg [7], Kaleva and Seikkala [15], Kramosil and Michalek [18] have introduced the concepts of fuzzy metric space in different ways. George and Veeramani [8, 9] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek and defined the Hausdorff topology of fuzzy metric spaces. They showed also that every metric induces a fuzzy metric.

Grabiec [10] extended the well known fixed point theorem of Banach and Edelstein [6] to fuzzy metric spaces in the sense of Kramosil and Michalek. Grabiec and Kramosil and Michalek, Mishra et al. [20] and many authors [11–14, 20, 22, 24] obtained common fixed point theorems for compatible maps and asymptotically commuting maps on fuzzy metric spaces, which generalize, extend and fuzzify several fixed point theorems for contractive-type maps on metric spaces and other spaces. Cho et al. [3] first formulate the definition of compatible maps of type ($\beta$) in fuzzy metric spaces and give some relations between the concepts of compatible maps and compatible

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maps of types $(\alpha)$ and $(\beta)$ and give some fixed point theorems for compatible maps of type $(\beta)$ on fuzzy metric spaces.

Atanassov [1] introduced and studied the the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [25]. Çoker [4] introduced the concepts of the so-called "intuitionistic fuzzy topological spaces". There has been much progress in the study of intuitionistic fuzzy sets by many authors [1, 2, 4–7, 10, 15, 21]. Park [21] using the idea of intuitionistic fuzzy sets, define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorm as a generalization of fuzzy metric space due to George and Veeramani and introduce the notion of Cauchy sequences in an intuitionistic fuzzy metric space and prove the Baire's theorem and finding a necessary and suffient condition for an intuitionistic fuzzy metric spaces to be complete and show that every separable intuitionistic fuzzy metric space is second countable and that every subspace of an intuitionistic fuzzy metric space is separable and prove the Uniform limit theorem for intuitionistic fuzzy metric spaces.

In this paper, we first formulate the definition of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$ in intuitionistic fuzzy metric spaces. Thereafter, we give some relations between the concepts of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$ in intuitionistic fuzzy metric spaces.

## 1. Intuitionistic fuzzy metric spaces

DEFINITION 1 ([23]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

**(a)** $*$ is commutative and associative;
**(b)** $*$ is continuous;
**(c)** $a * 1 = a$ for all $a \in [0,1]$;
**(d)** $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

DEFINITION 2 ([23]). A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ is satisfying the following conditions:

**(a)** $\Diamond$ is commutative and associative;
**(b)** $\Diamond$ is continuous;
**(c)** $a \Diamond 0 = a$ for all $a \in [0,1]$;
**(d)** $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Several examples and detals for the concepts of triangular norms (t-norms) and triangular conorms (t-conorms) were proposed by many authors (see [16, 17]).

DEFINITION 3 ([21]). A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and $M$, $N$ are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x$, $y$, $z \in X$, $s$, $t > 0$,

**(IFM-1)** $M(x, y, t) + N(x, y, t) \leq 1$;
**(IFM-2)** $M(x, y, t) > 0$;
**(IFM-3)** $M(x, y, t) = 1$ if and only if $x = y$;
**(IFM-4)** $M(x, y, t) = M(y, x, t)$;
**(IFM-5)** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
**(IFM-6)** $M(x, y, .) : [0, \infty) \to [0, 1]$ is continuous;
**(IFM-7)** $N(x, y, t) > 0$;
**(IFM-8)** $N(x, y, t) = 0$ if and only if $x = y$;
**(IFM-9)** $N(x, y, t) = N(y, x, t)$;
**(IFM-10)** $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$;
**(IFM-11)** $N(x, y, .) : [0, \infty) \to [0, 1]$ is continuous.

Then $(M, N)$ is called an intuitionistic fuzzy metric on $X$. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

REMARK 1. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are assosiated [19], i.e. $x \Diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0, 1]$.

REMARK 2. In intuitionistic fuzzy metric space $X$, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.

EXAMPLE 1 (Induced intuitionistic fuzzy metric [21]). Let $(X, d)$ be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let $M_d$ and $N_d$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{ht^n}{ht^n + md(x, y)}, \qquad N_d(x, y, t) = \frac{d(x, y)}{kt^n + md(x, y)}$$

for all $h, k, m, n \in \mathbb{R}^+$. Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space.

REMARK 3. Note the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$ and hence $(M, N)$ is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm. In the above example by taking $h = k = m = n = 1$, we get

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \qquad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

We call this intuitionistic fuzzy metric induced by a metric $d$ the standard intuitionistic fuzzy metric.

EXAMPLE 2 ([21]). Let $X = \mathbb{N}$. Define $a * b = \max\{0, a + b - 1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0, 1]$ and let $M$ and $N$ be fuzzy sets on $X^2 \times (0, \infty)$ as follows:

$$M(x,y,t) = \begin{cases} \dfrac{x}{y} \text{ if } x \leq y, \\ \dfrac{y}{x} \text{ if } y \leq x, \end{cases} \qquad N(x,y,t) = \begin{cases} \dfrac{y-x}{y} \text{ if } x \leq y, \\ \dfrac{x-y}{x} \text{ if } y \leq x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

REMARK 4. Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric $d$ on $X$ satisfying

$$M(x,y,t) = \frac{t}{t + d(x,y)}, \qquad N(x,y,t) = \frac{d(x,y)}{t + d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note the above functions $(M, N)$ is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$.

DEFINITION 4. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

**(a)** a sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if for each $t > 0$ and $p > 0$,

$$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0.$$

**(b)** a sequence $\{x_n\}$ in $X$ is converging to $x$ in $X$ if for each $t > 0$,

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x, t) = 0.$$

**(c)** An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

REMARK 5. Since $*$ and $\Diamond$ are continuous, the limit is uniquely determined from (IFM-5) and (IFM-10).

Throughout this paper, $(X, M, N, *, \Diamond)$ will denote the intuitionistic fuzzy metric space in the sense of Definition 3 with the following condition:

**(IFM-12)** $\lim_{t\to\infty} M(x,y,t) = 1$ for all $x, y \in X$ and $t > 0$;

**(IFM-13)** $\lim_{t\to\infty} N(x,y,t) = 0$ for all $x, y \in X$ and $t > 0$.

## 2. Main results

In this section, we introduce the concept of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$ and give some relations between the concepts of compatible maps and compatible maps of types $(\alpha)$ and $(\beta)$ in intuitionistic fuzzy metric space.

DEFINITION 5. Let $A$ and $B$ be maps from an intuitionistic fuzzy metric space (IFM-space) $(X, M, N, *, \Diamond)$ into itself. The maps $A$ and $B$ are said to be compatible if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

DEFINITION 6. Let $A$ and $B$ be maps from an IFM-space $(X, M, N, *, \Diamond)$ into itself. The maps $A$ and $B$ are said to be compatible of type $(\alpha)$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0,$$
$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

DEFINITION 7. Let $A$ and $B$ be maps from an IFM-space $(X, M, N, *, \Diamond)$ into itself. The maps $A$ and $B$ are said to be compatible of type $(\beta)$ if, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

PROPOSITION 1. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. Then $A$ and $B$ are compatible if and only if they are compatible maps of type $(\alpha)$.*

Proof. Suppose that $A$ and $B$ are compatible and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since $A$ and $B$ are continuous, we have

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az,$$
$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_n = Bz.$$

Further, since $A$ and $B$ are compatible,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$. Now, since we have

$$M(ABx_n, BBx_n, t) \geq M\left(ABx_n, BAx_n, \frac{t}{2}\right) * M\left(BAx_n, BBx_n, \frac{t}{2}\right)$$

and

$$N(ABx_n, BBx_n, t) \leq N\left(ABx_n, BAx_n, \frac{t}{2}\right) \Diamond N\left(BAx_n, BBx_n, \frac{t}{2}\right)$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) \leq 0 \Diamond 0 \leq 0$$

which implies that

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0.$$

Similarly, we have also, for all $t > 0$,

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible of type $(\alpha)$.

Conversely, suppose that $A$ and $B$ are compatible of type $(\alpha)$ and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since $A$ and $B$ are continuous, we have

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az,$$
$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_n = Bz.$$

Further, since $A$ and $B$ are compatible of type $(\alpha)$, we have, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0,$$
$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.$$

Thus, from the inequality

$$M(ABx_n, BAx_n, t) \geq M\left(ABx_n, BBx_n, \frac{t}{2}\right) * M\left(BBx_n, BAx_n, \frac{t}{2}\right)$$

and

$$N(ABx_n, BAx_n, t) \leq N\left(ABx_n, BBx_n, \frac{t}{2}\right) \Diamond N\left(BBx_n, BAx_n, \frac{t}{2}\right)$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) \leq 0 \Diamond 0 \leq 0$$

for all $t > 0$, which implies that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible. This completes the proof. ∎

PROPOSITION 2. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. Then $A$ and $B$ are compatible if and only if they are compatible maps of type $(\beta)$.*

Proof. Suppose that $A$ and $B$ are compatible and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since $A$ and $B$ are continuous, we have

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az,$$
$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_n = Bz.$$

Further, since $A$ and $B$ are compatible,

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0$$

for all $t > 0$. Now, since we have

$$\begin{aligned} M(AAx_n, BBx_n, t) &\geq M\left(AAx_n, ABx_n, \frac{t}{2}\right) * M\left(ABx_n, BBx_n, \frac{t}{2}\right) \\ &\geq M\left(AAx_n, ABx_n, \frac{t}{2}\right) * M\left(ABx_n, BAx_n, \frac{t}{4}\right) \\ &\quad * M\left(BAx_n, BBx_n, \frac{t}{4}\right) \end{aligned}$$

and

$$\begin{aligned} N(AAx_n, BBx_n, t) &\leq N\left(AAx_n, ABx_n, \frac{t}{2}\right) \Diamond N(ABx_n, BBx_n, \frac{t}{2}) \\ &\leq N\left(AAx_n, ABx_n, \frac{t}{2}\right) \Diamond N\left(ABx_n, BAx_n, \frac{t}{4}\right) \\ &\quad \Diamond M\left(BAx_n, BBx_n, \frac{t}{4}\right) \end{aligned}$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) \geq 1 * 1 * 1 \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(AAx_n, BBx_n, t) \leq 0 \Diamond 0 \Diamond 0 \leq 0 \Diamond 0 \leq 0$$

which implies that

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible maps of type $(\beta)$.

Conversely, suppose that $A$ and $B$ are compatible maps of type $(\beta)$ and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since $A$ and $B$ are continuous, we have

$$\begin{aligned} \lim_{n\to\infty} AAx_n &= \lim_{n\to\infty} ABx_n = Az, \\ \lim_{n\to\infty} BAx_n &= \lim_{n\to\infty} BBx_n = Bz. \end{aligned}$$

Further, since $A$ and $B$ are compatible maps of type $(\beta)$, we have, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Thus, from the inequality

$$\begin{aligned} M(ABx_n, BAx_n, t) &\geq M\left(ABx_n, AAx_n, \frac{t}{2}\right) * M\left(AAx_n, BAx_n, \frac{t}{2}\right) \\ &\geq M\left(ABx_n, AAx_n, \frac{t}{2}\right) * M\left(AAx_n, BBx_n, \frac{t}{4}\right) \\ &\quad * M\left(BBx_n, BAx_n, \frac{t}{4}\right) \end{aligned}$$

and

$$N(ABx_n, BAx_n, t) \leq N\left(ABx_n, AAx_n, \frac{t}{2}\right) \Diamond N\left(AAx_n, BAx_n, \frac{t}{2}\right)$$
$$\leq N\left(ABx_n, AAx_n, \frac{t}{2}\right) \Diamond N\left(AAx_n, BBx_n, \frac{t}{4}\right)$$
$$\Diamond M\left(BBx_n, BAx_n, \frac{t}{4}\right)$$

it follows that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) \geq 1 * 1 * 1 \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BAx_n, t) \leq 0 \Diamond 0 \Diamond 0 \leq 0 \Diamond 0 \leq 0$$

for all $t > 0$, which implies that

$$\lim_{n\to\infty} M(ABx_n, BAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BAx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible. This completes the proof. ∎

PROPOSITION 3. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ be compatible maps of type $(\alpha)$. If one of $A$ and $B$ is continuous, then $A$ and $B$ are compatible maps of type $(\beta)$.*

Proof. Suppose that $A$ and $B$ compatible maps of type $(\alpha)$ and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Assume, without loss of generality, that $A$ is continuous. Then

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az.$$

Further, since $A$ and $B$ are compatible maps of type $(\alpha)$, we have also, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0,$$
$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.$$

Thus, from the inequality

$$M(AAx_n, BBx_n, t) \geq M\left(AAx_n, ABx_n, \frac{t}{2}\right) * M\left(ABx_n, BBx_n, \frac{t}{2}\right)$$

and

$$N(AAx_n, BBx_n, t) \leq N\left(AAx_n, ABx_n, \frac{t}{2}\right) \Diamond N\left(ABx_n, BBx_n, \frac{t}{2}\right)$$

it follows that

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(AAx_n, BBx_n, t) \leq 0 \Diamond 0 \leq 0$$

which implies that

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible maps of type $(\beta)$. This completes the proof. ∎

PROPOSITION 4. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. If $A$ and $B$ are compatible maps of type $(\beta)$, then they are compatible maps of type $(\alpha)$.*

Proof. Suppose that $A$ and $B$ are compatible maps of type $(\beta)$ and let $\{x_n\}$ be a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Since $A$ and $B$ are continuous, we have

$$\lim_{n\to\infty} AAx_n = \lim_{n\to\infty} ABx_n = Az,$$
$$\lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_n = Bz.$$

Further, since $A$ and $B$ are compatible maps of type $(\beta)$, we have, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Thus, from the inequality

$$M(ABx_n, BBx_n, t) \geq M\left(ABx_n, AAx_n, \frac{t}{2}\right) * M\left(AAx_n, BBx_n, \frac{t}{2}\right)$$

and

$$N(ABx_n, BBx_n, t) \leq N\left(ABx_n, AAx_n, \frac{t}{2}\right) \Diamond N\left(AAx_n, BBx_n, \frac{t}{2}\right)$$

it follows that

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(ABx_n, BBx_n, t) \leq 0 \Diamond 0 \leq 0$$

which implies that, for all $t > 0$,

$$\lim_{n\to\infty} M(ABx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(ABx_n, BBx_n, t) = 0.$$

Similarly, we have also, for all $t > 0$,

$$\lim_{n\to\infty} M(BAx_n, AAx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(BAx_n, AAx_n, t) = 0.$$

Therefore, $A$ and $B$ are compatible maps of type $(\alpha)$. This completes the proof. ∎

From Proposition 3 and 4, we have the following.

PROPOSITION 5. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ be continuous maps from $X$ into itself. Then $A$ and $B$ are compatible maps of type $(\alpha)$ if and only if they are compatible maps of type $(\beta)$.*

PROPOSITION 6. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$ and let $A$ and $B$ maps from $X$ into itself. If $A$ and $B$ are compatible maps of type $(\beta)$ and $Az = Bz$ for some $z \in X$, then $ABz = BBz = BAz = AAz$.*

Proof. Suppose that $\{x_n\}$ is a sequence in $X$ defined by $x_n = z$ for some $z \in X$ and $n = 1, 2, 3...$and $Az = Bz$. Then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = Az.$$

Since $A$ and $B$ are compatible maps of type $(\beta)$, we have, for all $t > 0$,

$$M(AAz, BBz, t) = \lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1$$

and

$$N(AAz, BBz, t) = \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0$$

and so $AAz = BBz$. On the other hand, from $Az = Bz$, it follows that $AAz = ABz$ and $BAz = BBz$.

Therefore, we have $ABz = AAz = BBz = BAz$. This completes the proof. ∎

PROPOSITION 7. *Let $(X, M, N, *, \Diamond)$ be an IFM-space with $t * t \geq t$ and $(1-t)\Diamond(1-t) \leq (1-t)$ for all $t \in [0,1]$. Let $A$ and $B$ are compatible maps of type $(\beta)$ from $X$ into itself and let $\{x_n\}$ be a sequence in $X$ such that and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$. Then we have the following:*

**(i)** $\lim_{n\to\infty} BBx_n = Az$ if $A$ is continuous at $z$,

**(ii)** $\lim_{n\to\infty} AAx_n = Bz$ if $B$ is continuous at $z$,

**(iii)** $ABz = BAz$ and $Az = Bz$ if $A$ and $B$ are continuous at $z$.

Proof. **(i)** Suppose that $A$ is continuous at $z$. From $\lim_{n\to\infty} Ax_n = z$ for some $z \in X$, it follows that $\lim_{n\to\infty} AAx_n = Az$. Further, since $A$ and $B$ are compatible maps of type $(\beta)$, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Thus, from the inequality

$$M(BBx_n, Az, t) \geq M\left(BBx_n, AAx_n, \frac{t}{2}\right) * M\left(AAx_n, Az, \frac{t}{2}\right)$$

and

$$N(BBx_n, Az, t) \leq N\left(BBx_n, AAx_n, \frac{t}{2}\right) \Diamond N\left(AAx_n, Az, \frac{t}{2}\right)$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(BBx_n, Az, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(BBx_n, Az, t) \leq 0 \Diamond 0 \leq 0.$$

Therefore, we have $\lim_{n\to\infty} BBx_n = Az$.

**(ii)** Suppose that $B$ is continuous at $z$. From $\lim_{n\to\infty} Bx_n = z$ for some $z \in X$, it follows that $\lim_{n\to\infty} BBx_n = Bz$. Further, since $A$ and $B$ are compatible maps of type $(\beta)$, for all $t > 0$,

$$\lim_{n\to\infty} M(AAx_n, BBx_n, t) = 1 \text{ and } \lim_{n\to\infty} N(AAx_n, BBx_n, t) = 0.$$

Thus, from the inequality

$$M(AAx_n, Bz, t) \geq M\left(AAx_n, BBx_n, \frac{t}{2}\right) * M\left(BBx_n, Bz, \frac{t}{2}\right)$$

and

$$N(AAx_n, Bz, t) \leq N\left(AAx_n, BBx_n, \frac{t}{2}\right) \Diamond N\left(BBx_n, Bz, \frac{t}{2}\right)$$

for all $t > 0$, it follows that

$$\lim_{n\to\infty} M(AAx_n, Bz, t) \geq 1 * 1 \geq 1$$

and

$$\lim_{n\to\infty} N(AAx_n, Bz, t) \leq 0 \Diamond 0 \leq 0.$$

Therefore, we have $\lim_{n\to\infty} AAx_n = Bz$.

**(iii)** Suppose that $A$ and $B$ is continuous at $z$. Since $A$ is continuous at $z$, by (i), we have $\lim_{n\to\infty} BBx_n = Az$. On the other hand, since $\lim_{n\to\infty} Bx_n = z$ for some $z \in X$ and $B$ is continuous at $z$, $\lim_{n\to\infty} AAx_n = Bz$. Therefore, by the uniqueness of the limit, we have $Az = Bz$ and so, by Proposition 6 it follows that $ABz = BAz$. This completes the proof. ∎

## References


[1] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87–96.

[2] K. Atanassov, *New operations defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems 61 (1994), 137–142.

[3] Y. J. Cho, H. K. Pathak, S. M. Kang and J. S. Jung, *Common fixed points of compatible maps of type ($\beta$) on fuzzy metric spaces*, Fuzzy Sets and Systems 93 (1998), 99–111.

[4] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy Sets and Systems 88 (1997), 81–89.

[5] Z. K. Deng, *Fuzzy pseudo-metric spaces*, J. Math. Anal. Appl. 86 (1982), 74–95.

[6] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), 74–79.

[7] M. A. Erceg, *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl. 69 (1979), 205–230.

[8] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems 64 (1994), 395–399.

[9] A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces*, Fuzzy Sets and Systems 90 (1997), 365–368.

[10] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems 27 (1988), 385–389.

[11] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly. 83 (1976), 261–263.

[12] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Sci. 9 (1986),771–779.

[13] G. Jungck, P. P. Murthy, and Y. J. Cho, *Compatible mapping of type (A) and common fixed points*, Math. Japonica 38 (1993), 381–390.

[14] G. Jungck and B. E. Rhoades, *Some fixed point theorems for compatible maps*, Internat. J. Math. & Math. Sci. 3 (1993), 417–428.

[15] O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems 12 (1984), 225–229.

[16] E. P. Klement, R. Mesiar and E. Pap, *A characterization of the ordering of continuous t-norms*, Fuzzy Sets and Systems 86 (1997), 189–195.

[17] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Kluwer Academic Pub. Trends in Logic 8, Dordrecht 2000.

[18] O. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica 11 (1975), 326–334.

[19] R. Lowen, *Fuzzy Set Theory*, Kluwer Academic Pub., Dordrecht 1996.

[20] S. N. Mishra, N. Sharma and S. L. Singh, *Common fixed points of maps on fuzzy metric spaces*, Internat. J. Math. & Math. Sci. 17 (1994), 253–258.

[21] J. H. Park, *Intuitionistic fuzzy metric spaces*, Chaos, Solitons & Fractals 22 (2004), 1039–1046.
[22] H. K. Pathak, Y. J. Cho, J. M. Kang and B. Madharia, *Compatible mappings of type (C) and common fixed point theorems of Gregus type*, Demonstratio Math. Vol. 31 (3) (1998), 499–518.
[23] B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math. 10 (1960), 314–334.
[24] S. Sharma and B. Deshpande, *Common fixed points of compatible maps of type ($\beta$) on fuzzy metric spaces*, Demonstratio Math. Vol. 35 (1) (2002), 165–174.
[25] L. A. Zadeh, *Fuzzy sets*, Inform and Control 8 (1965), 338–353.

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